

# Characterizations of all-derivable points in $B(H)^1$

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## Abstract

Let  $\mathcal{K}$  and  $\mathcal{H}$  be two Hilbert space, and let  $B(\mathcal{K}, \mathcal{H})$  be the algebra of all bounded linear operators from  $\mathcal{K}$  into  $\mathcal{H}$ . We say that an element  $G \in B(\mathcal{H}, \mathcal{H})$  is an all-derivable point in  $B(\mathcal{H}, \mathcal{H})$  if every derivable linear mapping  $\varphi$  at  $G$  (i.e.  $\varphi(ST) = \varphi(S)T + S\varphi(T)$  for any  $S, T \in B(H)$  with  $ST = G$ ) is a derivation. Let both  $\varphi : B(\mathcal{H}, \mathcal{K}) \rightarrow B(\mathcal{H}, \mathcal{K})$  and  $\psi : B(\mathcal{K}, \mathcal{H}) \rightarrow B(\mathcal{K}, \mathcal{H})$  be two linear mappings. In this paper, the following results will be proved : if  $Y\varphi(W) = \psi(Y)W$  for any  $Y \in B(\mathcal{K}, \mathcal{H})$  and  $W \in B(\mathcal{H}, \mathcal{K})$ , then  $\varphi(W) = DW$  and  $\psi(Y) = YD$  for some  $D \in B(\mathcal{K})$ . As an important application, we will show that an operator  $G$  is an all-derivable point in  $B(\mathcal{H}, \mathcal{H})$  if and only if  $G \neq 0$ .

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## 1. Introduction

Let  $\mathcal{A}$  be an algebra, and let  $\varphi$  be a linear mappings on  $\mathcal{A}$ . We say that  $\varphi$  is a derivation if  $\varphi(ST) = \varphi(S)T + S\varphi(T)$  for any  $S, T \in \mathcal{A}$ . Fix an operator  $G \in \mathcal{A}$ . We say that  $\varphi$  is a derivable mapping at  $G$  if  $\varphi(ST) = \varphi(S)T + S\varphi(T)$  for any  $S, T \in \mathcal{A}$  with  $ST = G$ . An element  $G \in \mathcal{A}$  is called an all-derivable point in  $\mathcal{A}$  if every derivable mapping at  $G$  is a derivation.

We describe some of the results related to ours. Jin, Lu and Li [2] show that every derivable mapping  $\varphi$  at 0 with  $\varphi(I) = 0$  on nest algebras is a derivation. Hou and Qi [3] prove that every derivable mapping at the unit operator on J-subspace lattice algebras is a derivation. Zhu and Zhao [8] give the characterizations of all-derivable points in nest algebras  $alg\mathcal{N}$  with nontrivial nest  $\mathcal{N}$  (on Hilbert spaces), i.e.  $G \in alg\mathcal{N}$  is an all-derivable point if and only if  $G \neq 0$ . But the condition whether the assumption of "nontrivial" may be omitted, remains open. It is obvious to see that a nontrivial nest algebra is essentially a triangular algebra, but trivial nest algebras  $B(\mathcal{H})$  is not triangular algebra, this case is more challenging than of triangular algebras.

Recently, Zhang, Hou and Qi [7] have proved the following theorem:

**Theorem 1.1** [7] *Let  $\mathcal{N}$  be a complete nest on a complex Banach space  $\mathcal{X}$  with  $\dim\mathcal{X} \geq 2$ , and  $\delta : alg\mathcal{N} \rightarrow alg\mathcal{N}$  be a linear mapping. Let  $Z \in alg\mathcal{N}$  be an injective operator or an operator with dense range in  $alg\mathcal{N}$ . Then  $\delta$  is derivable at the operator  $Z$  if and only if  $\delta$  is a derivation. That is, every injective operator and every operator with dense range are all-derivable points of any nest algebra.*

The above result implies the following corollary and it provides a basis for us to solve the above problem.

**Corollary 1.2** *Every injective operator and every operator with dense range are all-derivable points of a nest algebra( on Hilbert spaces).*

The purpose of the present paper is to solve the above problem and prove that  $G \in B(\mathcal{H})$  is an all-derivable point if and only if  $G \neq 0$ . Furthermore, we obtain that  $G \in alg\mathcal{N}$  is an all-derivable point if and only if  $G \neq 0$  in any nest algebra. For other results, see [1,4,5].

This paper is organized as follows. In Section 2, we describe the major Theorem 2.1 in this paper, and give a preliminary Theorem 2.2 and its proof. Using the results, we give the proof of our main Theorem 2.1 in Section 3, that is  $G$  is an all-derivable point if and only if  $G \neq 0$  in any nest algebras.

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## 2. The main theorem and a lemma

In this section we fix some notations. Let  $\mathcal{H}$  and  $\mathcal{K}$  be two Hilbert spaces. We use the symbols  $B(\mathcal{K}, \mathcal{H})$  and  $F(\mathcal{K}, \mathcal{H})$  denote the set of all linear bounded operators from  $\mathcal{K}$  into  $\mathcal{H}$  and the set of all finite rank operators from  $\mathcal{K}$  into  $\mathcal{H}$ , respectively. The symbols  $B(\mathcal{H}, \mathcal{H})$  and  $F(\mathcal{H}, \mathcal{H})$  is abbreviated to  $B(\mathcal{H})$  and  $F(\mathcal{H})$ , respectively. Let  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ , we use the symbols  $x \otimes y \in B(\mathcal{K}, \mathcal{H})$  and  $I_{\mathcal{H}} \in B(\mathcal{H})$  denote the rank one operator  $\langle \cdot, y \rangle x$  and the unit operator on  $\mathcal{H}$ , respectively. Assume that  $Y \in B(\mathcal{K}, \mathcal{H})$ , we always denote the range and kernel of  $Y$  by the symbols  $\text{ran}Y$  and  $\text{ker}Y$ , respectively. we write  $\mathbf{C}$  for the complex number field.

The following theorem is our main result:

**Theorem 2.1** *Let  $\mathcal{H}$  be a Hilbert space and  $G \in B(\mathcal{H})$ . Then  $G$  is an all-derivable point in  $B(\mathcal{H})$  if and only if  $G \neq 0$ .*

The following theorem will play an important role for our purposes.

**Theorem 2.2** *Let  $\mathcal{H}$  and  $\mathcal{K}$  be two Hilbert spaces, and let  $\varphi : B(\mathcal{H}, \mathcal{K}) \rightarrow B(\mathcal{H}, \mathcal{K})$  and  $\psi : B(\mathcal{K}, \mathcal{H}) \rightarrow B(\mathcal{K}, \mathcal{H})$  be two linear mappings. If the following equation holds*

$$Y\varphi(W) = \psi(Y)W \quad (1)$$

for any  $Y \in B(\mathcal{K}, \mathcal{H})$  and  $W \in B(\mathcal{H}, \mathcal{K})$ , then there exists an operator  $D \in B(\mathcal{K})$  such that

$$\varphi(W) = DW \quad \text{and} \quad \psi(Y) = YD$$

for any  $Y \in B(\mathcal{K}, \mathcal{H})$  and  $W \in B(\mathcal{H}, \mathcal{K})$ .

**Proof.** Step 1. Fix an operator  $W \in B(\mathcal{H}, \mathcal{K})$  with  $\text{ran}W = \overline{\text{ran}W}^{\|\cdot\|} = \mathcal{K}_1 \subseteq \mathcal{K}$ . By the equation (1), we have

$$\psi(x \otimes y)\mathcal{K}_1 = \psi(x \otimes y)\text{ran}(W) \subseteq \text{ran}(x \otimes y) = \mathbf{C}x$$

for any  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ . Thus there exists a continuous linear functional  $\lambda_{y,W}$  on  $\mathcal{K}_1$  such that

$$\psi(x \otimes y)(u) = \lambda_{y,W}(u)x$$

for any  $u \in \mathcal{K}_1$ . By Riesz representation theorem, there exists a vector  $A_W(y) \in \mathcal{K}_1$  such that  $\lambda_{y,W}(u) = \langle u, A_W(y) \rangle$  and

$$\psi(x \otimes y)(u) = \langle u, A_W(y) \rangle x = x \otimes A_W(y)(u)$$

for any  $u \in \mathcal{K}_1$ , i.e.  $\psi(x \otimes y) = x \otimes A_W(y)$ . By equation (1) and the above equation, we have

$$(x \otimes y)\varphi(W) = \psi(x \otimes y)W = x \otimes A_W(y)W. \quad (2)$$

It is easy to verify from the linear of  $\psi$  that  $A_W : \mathcal{K} \rightarrow \mathcal{K}_1$  is a linear operator, and

$$\begin{aligned} & \|\langle Wv, A_W(y) \rangle\| x\| = \|(x \otimes A_W(y))Wv\| = \|\psi(x \otimes y)Wv\| \\ & = \|\psi(x \otimes y)\varphi(W)v\| \leq \|x\| \|y\| \|\varphi(W)v\| \end{aligned}$$

for any  $y, v \in \mathcal{K}$ . Fix a vector  $u \in \mathcal{K}_1 = \text{ran}W$ , then the set  $\{\langle A_W(y), u \rangle : y \in \mathcal{K}_1, \|y\| \leq 1\}$  is a bounded set. By the uniform boundedness principle,  $\{\|A_W(y)\| : y \in \mathcal{K}_1, \|y\| \leq 1\}$  is a bounded set, i.e.  $A_W \in B(\mathcal{K}, \mathcal{K}_1)$ . It follows from the equation (2) that  $(x \otimes y)\varphi(W) = x \otimes A_W(y)W = x \otimes yA_W^*W$ , or

$$\varphi(W) = A_W^*W. \quad (3)$$

Step 2. For any  $x \in \mathcal{H}$  and  $y \in \mathcal{K}$ , then  $y \otimes x \in B(\mathcal{H}, \mathcal{K})$  and  $\text{ran}(y \otimes x) = \overline{\text{ran}(y \otimes x)}^{\|\cdot\|} = \mathbf{C}y$ . It follows from equation (3) that  $\varphi(y \otimes x) = A_{y \otimes x}(y \otimes x)$ . Define a mapping  $B_y : \mathcal{H} \rightarrow \mathcal{H}$  as follows:

$$B_x y = A_{y \otimes x}^* y$$

for any  $y \in \mathcal{K}$ . Then we have

$$\varphi(y \otimes x) = B_x y \otimes x$$

for any  $y \in \mathcal{K}$ . It is easy to verify from the above equation that  $B_x$  is a linear mapping on  $\mathcal{K}$ . We claim that  $B_x$  is independent of  $x$ . In fact, for any  $x_1, x_2 \in \mathcal{H}$ , we have

$$\varphi(y \otimes x_i) = B_{x_i} y \otimes x_i \quad (i = 1, 2);$$

and

$$\varphi(y \otimes (x_1 + x_2)) = B_{x_1+x_2} y \otimes (x_1 + x_2).$$

Combining the above three equations, we obtain

$$(B_{x_1} - B_{x_1+x_2})y \otimes x_1 + (B_{x_2} - B_{x_1+x_2})y \otimes x_2 = 0$$

If  $x_1$  and  $x_2$  are linearly independent, then  $B_{x_1}y = B_{x_1+x_2}y = B_{x_2}y$  for any  $y \in \mathcal{K}$ , i.e.  $B_{x_1} = B_{x_2}$ . If  $x_2 = \alpha x_1$ , then

$$B_{x_2}y \otimes x_2 = \varphi(y \otimes x_2) = \varphi(y \otimes \alpha x_1) = \bar{\alpha}\varphi(y \otimes x_1) = \bar{\alpha}B_{x_1}y \otimes x_1 = B_{x_1}y \otimes \alpha x_1 = B_{x_1}y \otimes x_2$$

for any  $y \in \mathcal{K}$ . Thus  $B_{x_2} = B_{x_1}$ . This implies that  $B_x$  is independent of  $x$ . So we may write  $D = B_x$  and get

$$\varphi(y \otimes x) = D(y \otimes x)$$

for any  $y \in \mathcal{K}$  and  $x \in \mathcal{H}$ . Hence

$$\varphi(F) = DF \tag{4}$$

for any  $F \in F(\mathcal{H}, \mathcal{K})$ .

Step 3. Define a mapping  $\Phi : B(\mathcal{H}, \mathcal{K}) \rightarrow B(\mathcal{H}, \mathcal{K})$  as follows:

$$\Phi(W) = \varphi(W) - DW$$

for any  $W \in B(\mathcal{H}, \mathcal{K})$ . We claim that  $\Phi \equiv 0$ . Otherwise, there exists  $W \in B(\mathcal{H}, \mathcal{K})$  such that  $\Phi(W) \neq 0$ . Furthermore we can find a vector  $x \in \mathcal{H}$  with  $\|x\| = 1$  such that  $y = \Phi(W)x \neq 0$ . Thus we have

$$\begin{aligned} & (x \otimes y)\Phi(W(I - x \otimes x))(x \otimes x) \\ &= (x \otimes y)[\varphi(W(I - x \otimes x)) - DW(I - x \otimes x)](x \otimes x) \\ &= \psi(x \otimes y)W(I - x \otimes x)(x \otimes x) = 0. \end{aligned}$$

Note that  $\Phi(F) = 0$  for any  $F \in F(\mathcal{H}, \mathcal{K})$ , so we have

$$\begin{aligned} 0 &= (x \otimes y)\Phi(W(I - x \otimes x))(x \otimes x) = (x \otimes y)[\Phi(W) - \Phi(Wx \otimes x)](x \otimes x) \\ &= (x \otimes y)\Phi(W)(x \otimes x) = \langle \Phi(W)x, y \rangle x \otimes x = \|y\|^2 x \otimes x \end{aligned}$$

This is a contradiction with  $x \neq 0$  and  $y \neq 0$ . Hence  $\Phi \equiv 0$ , i.e.  $\varphi(W) = DW$  for any  $W \in B(\mathcal{H}, \mathcal{K})$ . Furthermore, by the equation (1), we obtain  $\psi(Y)W = Y\varphi(W) = YDW$  for any  $Y \in B(\mathcal{K}, \mathcal{H})$  and  $W \in B(\mathcal{H}, \mathcal{K})$ . Hence

$$\psi(Y) = YD$$

for any  $Y \in B(\mathcal{K}, \mathcal{H})$ . This completes the proof of the theorem.

**Lemma 2.3.** Let  $\mathcal{A}$  be an operator subalgebra with unit operator  $I$  in  $B(H)$ , and let  $\varphi$  is a linear mapping from  $\mathcal{A}$  into  $B(\mathcal{K}_1, \mathcal{K}_2)$ . If  $\varphi(X) = 0$  for any invertible operator  $X \in \mathcal{A}$ , then  $\varphi \equiv 0$ .

**Proof.** For arbitrary operator  $X \in \mathcal{A}$ , there exists a real number  $\lambda > \|X\|$ . Then both  $\lambda I - X$  and  $2\lambda I - X$  are two invertible operators. So  $\varphi(\lambda I - X) = 0$  and  $\varphi(2\lambda I - X) = 0$ . It follows from the linearity of  $\varphi$  that  $\varphi(X) = 0$ . Hence  $\varphi \equiv 0$ . This completes the proof of the Lemma.  $\square$

### 3. The proof of Theorem 2.1

Now we will prove our main Theorem 2.1.

**The proof of Theorem 2.1.** Necessary. Suppose that  $G$  is an all-derivable point in  $B(\mathcal{H})$ . We claim that  $G \neq 0$ . In fact, the identity mapping  $\varphi$  on  $B(\mathcal{H})$  is a derivable mapping at 0, but  $\varphi$  is not derivation.

Sufficiency. Fix an operator  $0 \neq G \in B(\mathcal{H})$  and  $\varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  be a derivable mapping at  $G$ . We only need to prove that  $\varphi$  is a derivation. If  $\overline{\text{ran}G} = \mathcal{H}$ , then  $\varphi$  is a derivation by Corollary 1.2. If  $\overline{\text{ran}G} \neq \mathcal{H}$ , then we take  $\mathcal{H}_1 = \overline{\text{ran}G}$  and  $\mathcal{H}_2 = (\text{ran}G)^\perp$ . Obviously  $\dim \mathcal{H}_i \geq 1 (i = 1, 2)$ . Then  $G$  can be represented as a  $2 \times 2$  operator matrix relative to the orthogonal decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  as follows:

$$G = \begin{bmatrix} E & F \\ 0 & 0 \end{bmatrix},$$

where  $E \in B(\mathcal{H}_1)$ ,  $F \in B(\mathcal{H}_2, \mathcal{H}_1)$  with  $E \neq 0$  or  $F \neq 0$ . In the rest part of this paper, all  $2 \times 2$  operator matrixes are always represented as relative to the orthogonal decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ . For arbitrary operator  $S \in B(\mathcal{H})$ ,  $S$  can be expressed as the following operator matrix in the orthogonal decomposition of  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  as follows

$$S = \begin{bmatrix} X & Y \\ Z & Q \end{bmatrix},$$

where  $X \in B(\mathcal{H}_1)$ ,  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ ,  $Z \in B(\mathcal{H}_1, \mathcal{H}_2)$  and  $Q \in B(\mathcal{H}_2)$ . Since  $\varphi$  is a linear mapping, we can write

$$\left\{ \begin{array}{l} \varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) \\ A_{21}(X) & A_{22}(X) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} B_{11}(Y) & B_{12}(Y) \\ B_{21}(Y) & B_{22}(Y) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}\right) = \begin{bmatrix} C_{11}(Z) & C_{12}(Z) \\ C_{21}(Z) & C_{22}(Z) \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}\right) = \begin{bmatrix} D_{11}(Q) & D_{12}(Q) \\ D_{21}(Q) & D_{22}(Q) \end{bmatrix}, \end{array} \right.$$

where  $A_{ij}, B_{ij}, C_{ij}$  and  $D_{ij}$  are linear mappings from  $\mathcal{H}_j$  into  $\mathcal{H}_i$ . For arbitrary  $S, T \in B(\mathcal{H})$  with  $ST = G$ , we may write

$$S = \begin{bmatrix} X & Y \\ Z & Q \end{bmatrix}$$

and

$$T = \begin{bmatrix} U & V \\ W & R \end{bmatrix}.$$

It follows from  $ST = G$  that  $XU + YW = E$ ,  $XV + YR = F$ ,  $ZU + QW = 0$  and  $ZV + QR = 0$ . Since  $\varphi$  is a derivable mapping at  $G$ , we have

$$\begin{aligned} & \begin{bmatrix} A_{11}(E) + B_{11}(F) & A_{12}(E) + B_{12}(F) \\ A_{21}(E) + A_{21}(F) & A_{22}(E) + B_{22}(F) \end{bmatrix} = \varphi(G) = \varphi(S)T + S\varphi(T) \\ &= \begin{bmatrix} A_{11}(X) + B_{11}(Y) + C_{11}(Z) + D_{11}(Q) & A_{12}(X) + B_{12}(Y) + C_{12}(Z) + D_{12}(Q) \\ A_{21}(X) + B_{21}(Y) + C_{21}(Z) + D_{21}(Q) & A_{22}(X) + B_{22}(Y) + C_{22}(Z) + D_{22}(Q) \end{bmatrix} \begin{bmatrix} U & V \\ W & R \end{bmatrix} \\ &+ \begin{bmatrix} X & Y \\ Z & Q \end{bmatrix} \begin{bmatrix} A_{11}(U) + B_{11}(V) + C_{11}(W) + D_{11}(R) & A_{12}(U) + B_{12}(V) + C_{12}(W) + D_{12}(R) \\ A_{21}(U) + B_{21}(V) + C_{21}(W) + D_{21}(R) & A_{22}(U) + B_{22}(V) + C_{22}(W) + D_{22}(R) \end{bmatrix}. \end{aligned}$$

The above equation implies the following four operator equations

$$\begin{aligned}
& A_{11}(E) + B_{11}(F) \\
= & A_{11}(X)U + B_{11}(Y)U + C_{11}(Z)U + D_{11}(Q)U \\
& + A_{12}(X)W + B_{12}(Y)W + C_{12}(Z)W + D_{12}(Q)W \\
& + XA_{11}(U) + XB_{11}(V) + XC_{11}(W) + XD_{11}(R) \\
& + YA_{21}(U) + YB_{21}(V) + YC_{21}(W) + YD_{21}(R);
\end{aligned} \tag{5}$$

$$\begin{aligned}
& A_{12}(E) + B_{12}(F) \\
= & A_{11}(X)V + B_{11}(Y)V + C_{11}(Z)V + D_{11}(Q)V \\
& + A_{12}(X)R + B_{12}(Y)R + C_{12}(Z)R + D_{12}(Q)R \\
& + XA_{12}(U) + XB_{12}(V) + XC_{12}(W) + XD_{12}(R) \\
& + YA_{22}(U) + YB_{22}(V) + YC_{22}(W) + YD_{22}(R);
\end{aligned} \tag{6}$$

$$\begin{aligned}
& A_{21}(E) + B_{21}(F) \\
= & A_{21}(X)U + B_{21}(Y)U + C_{21}(Z)U + D_{21}(Q)U \\
& + A_{22}(X)W + B_{22}(Y)W + C_{22}(Z)W + D_{22}(Q)W \\
& + ZA_{11}(U) + ZB_{11}(V) + ZC_{11}(W) + ZD_{11}(R) \\
& + QA_{21}(U) + QB_{21}(V) + QC_{21}(W) + QD_{21}(R);
\end{aligned} \tag{7}$$

$$\begin{aligned}
& A_{22}(E) + B_{22}(F) \\
= & A_{21}(X)V + B_{21}(Y)V + C_{21}(Z)V + D_{21}(Q)V \\
& + A_{22}(X)R + B_{22}(Y)R + C_{22}(Z)R + D_{22}(Q)R \\
& + ZA_{12}(U) + ZB_{12}(V) + ZC_{12}(W) + ZD_{12}(R) \\
& + QA_{22}(U) + QB_{22}(V) + QC_{22}(W) + QD_{22}(R);
\end{aligned} \tag{8}$$

Note that the equations (5)-(8) always hold when  $XU + YW = E$ ,  $XV + YR = F$ ,  $ZU + QW = 0$  and  $ZV + QR = 0$ . Now we divide the proof of the theorem into the following eight Steps.

Step 1. We claim that  $A_{22} \equiv 0$ ,  $B_{21} \equiv 0$ ,  $C_{12} \equiv 0$ ,  $D_{22} \equiv 0$ ,  $A_{11}(I_{\mathcal{H}_1})F = 0$  and  $A_{12}(YW) = YC_{22}(W)$  for any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$  and  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ .

1) For any invertible operator  $X \in B(\mathcal{H}_1)$  and any  $\lambda \in \mathcal{C}$ , taking  $Y = \lambda F$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = X^{-1}E$ ,  $V = 0$ ,  $W = 0$  and  $R = \lambda^{-1}I_{\mathcal{H}_2}$  in the equation (8), then we have

$$A_{22}(E) + B_{22}(F) = \lambda^{-1}A_{22}(X) + B_{22}(F).$$

So  $A_{22}(X) = 0$ . By Lemma 2.3, we obtain  $A_{22} \equiv 0$ .

2) For any operator  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E$ ,  $V = F$ ,  $W = 0$  and  $R = 0$  in the equation (7) and (8) respectively, then we have

$$A_{21}(E) + B_{21}(F) = A_{21}(I_{\mathcal{H}_2})E + B_{21}(Y)E$$

and

$$A_{22}(E) + B_{22}(F) = A_{21}(I_{\mathcal{H}_2})F + B_{21}(Y)F.$$

In the rest part of the proof, we always assume that  $\lambda$  and  $\mu$  are two arbitrary nonzero real number. Replacing  $Y$  by  $\lambda Y$  in the above two equations, we have

$$A_{21}(E) + B_{21}(F) = A_{21}(I_{\mathcal{H}_2})E + \lambda B_{21}(Y)E$$

and

$$A_{22}(E) + B_{22}(F) = A_{21}(I_{\mathcal{H}_2})F + \lambda B_{21}(Y)F.$$

The above two equations implies that  $B_{21}(Y)E = 0$  and  $B_{21}(Y)F = 0$ . It follows from  $G = \begin{bmatrix} E & F \\ 0 & 0 \end{bmatrix}$  that  $B_{21}(Y) = B_{21}(Y) |_{\mathcal{H}_2} = B_{21}(Y) |_{\text{ran} G} = 0$ . Hence  $B_{21} \equiv 0$ .

3) For any invertible operator  $R \in B(\mathcal{H}_2)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Y = FR^{-1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E$ ,  $V = 0$  and  $W = 0$  in the equation (5), then we have

$$B_{11}(F) = A_{11}(I_{\mathcal{H}_1})E + B_{11}(FR^{-1})E + D_{11}(R) + FR^{-1}A_{21}(E) + FR^{-1}D_{21}(R).$$

Replacing  $R$  by  $\lambda R$  in the above equation, we have

$$B_{11}(F) = A_{11}(I_{\mathcal{H}_1})E + FR^{-1}D_{21}(R) + \lambda^{-1}[B_{11}(FR^{-1})E + FR^{-1}A_{21}(E)] + \lambda D_{11}(R).$$

Hence we have

$$B_{11}(F) = A_{11}(I_{\mathcal{H}_1})E + FR^{-1}D_{21}(R) \quad (9)$$

and  $D_{11}(R) = 0$ . By Lemma 2.3,  $D_{11} \equiv 0$ .

4) For any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E - YW$ ,  $V = F$  and  $R = 0$  in the equation (6), and noting that  $A_{22} \equiv 0$ , then we have

$$0 = A_{11}(I_{\mathcal{H}_1})F + B_{11}(Y)F - A_{12}(YW) + C_{12}(W) + YB_{22}(F) + YC_{22}(W). \quad (10)$$

Replacing  $W$  by  $\lambda W$  and taking  $Y = 0$  in the above equation (10), we have

$$0 = A_{11}(I_{\mathcal{H}_1})F + \lambda C_{12}(W).$$

So  $C_{12}(W) = 0$  for any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ , i.e.  $C_{12} \equiv 0$ . On the other hand, replacing  $W$  and  $Y$  by  $\lambda W$  and  $\lambda Y$  in the equation (10), respectively, and Noting that  $C_{12} \equiv 0$ , then we have

$$0 = A_{11}(I_{\mathcal{H}_1})F + \lambda[B_{11}(Y)F + YB_{22}(F)] + \lambda^2[YC_{22}(W) - A_{12}(YW)].$$

Thus we have

$$A_{11}(I_{\mathcal{H}_1})F = 0 \quad (11)$$

and

$$A_{12}(YW) = YC_{22}(W) \quad (12)$$

for any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$  and  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ .

Step 2. We claim that  $D_{22}$  is a derivation and  $D_{12}(R) = -A_{12}(I_{\mathcal{H}_1})R$  for any  $R \in B(\mathcal{H}_2)$ .

For any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and any  $R \in B(\mathcal{H}_2)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E$ ,  $V = F - YR$  and  $W = 0$  in the equation (6), and using the results in Step 1, then we have

$$\begin{aligned} A_{12}(E) + B_{12}(F) &= -A_{11}(I_{\mathcal{H}_1})YR + B_{11}(Y)F - B_{11}(Y)YR \\ &\quad + A_{12}(I_{\mathcal{H}_1})R + B_{12}(Y)R + A_{12}(E) \\ &\quad + B_{12}(F) - B_{12}(YR) + D_{12}(R) \\ &\quad + YB_{22}(F) - YB_{22}(YR) + YD_{22}(R). \end{aligned}$$

Replacing  $Y$  and  $R$  by  $\lambda Y$  and  $\mu R$  in the above equation, respectively, then we obtain

$$\begin{aligned} 0 &= \lambda[B_{11}(Y)F + YB_{22}(F)] + \lambda\mu[YD_{22}(R) + B_{12}(Y)R - A_{11}(I_{\mathcal{H}_1})YR \\ &\quad - B_{12}(YR)] - \lambda^2\mu[B_{11}(Y)YR - YB_{22}(YR)] + \mu[A_{12}(I_{\mathcal{H}_1})R + D_{12}(R)]. \end{aligned}$$

The above equation implies that

$$B_{12}(YR) = B_{12}(Y)R + YD_{22}(R) - A_{11}(I_{\mathcal{H}_1})YR \quad (13)$$

and

$$D_{12}(R) = -A_{12}(I_{\mathcal{H}_1})R \quad (14)$$

for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $R \in B(\mathcal{H}_2)$ .

We claim that  $A_{11}(I_{\mathcal{H}_1}) = 0$ . In fact, for any  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ , taking  $X = \lambda I_{\mathcal{H}_1}$ ,  $Y = \mu(F - \lambda V)$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = \lambda^{-1}E$ ,  $V = F - YR$ ,  $W = 0$  and  $R = \mu^{-1}I_{\mathcal{H}_2}$  in the equation (6), then we have

$$\begin{aligned} 0 &= \lambda[A_{11}(I_{\mathcal{H}_1})V - VD_{22}(I_{\mathcal{H}_2})] + \mu[B_{11}(F)V + FB_{22}(V)] \\ &\quad - \lambda\mu[B_{11}(V)V + VB_{22}(V)] + \lambda\mu^{-1}[A_{12}(I_{\mathcal{H}_1}) + D_{12}(I_{\mathcal{H}_2})] \\ &\quad + FD_{22}(I_{\mathcal{H}_2}). \end{aligned}$$

Thus we have

$$A_{11}(I_{\mathcal{H}_1})V = VD_{22}(I_{\mathcal{H}_2})$$

for any  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ . The above equation implies that  $A_{11}(I_{\mathcal{H}_1})(x \otimes y) = (x \otimes y)D_{22}(I_{\mathcal{H}_2})$  for any  $x \in \mathcal{H}_1$  and  $y \in \mathcal{H}_2$ . It follows that  $A_{11}(I_{\mathcal{H}_1})x = \lambda_x x$ . Hence  $A_{11}(I_{\mathcal{H}_1}) = \gamma I_{\mathcal{H}_1}$  for some  $\gamma \in \mathbb{C}$ . We only need to prove that  $\gamma = 0$ . Otherwise,  $\gamma \neq 0$ . By the equation (9) in Step 1, we have

$$B_{11}(F) = A_{11}(I_{\mathcal{H}_1})E + FR^{-1}D_{21}(R) = \gamma E + FR^{-1}D_{21}(R).$$

Supposing that  $F = 0$ , it follows from the above equation that  $E = 0$ . This is contradiction with  $G \neq 0$ . Supposing  $F \neq 0$ . Then  $A_{11}(I_{\mathcal{H}_1})F = \gamma F \neq 0$ . This is contradiction with the equation (11) in Step 1. Hence  $A_{11}(I_{\mathcal{H}_1}) = 0$ .

Now we show that  $D_{22}$  is a derivation. For any  $R_1, R_2 \in B(\mathcal{H}_2)$  and  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ , It follows from the equation (13) and  $A_{11}(I_{\mathcal{H}_1}) = 0$  that

$$B_{12}(YR_1R_2) = B_{12}(Y)R_1R_2 + YD_{22}(R_1R_2)$$

and

$$B_{12}(YR_1R_2) = B_{12}(YR_1)R_2 + YR_1D_{22}(R_2) = B_{12}(Y)R_1R_2 + YD_{22}(R_1)R_2 + YR_1D_{22}(R_2).$$

The above two equations implies that  $Y[D_{22}(R_1R_2) - D_{22}(R_1)R_2 - R_1D_{22}(R_2)] = 0$ , i.e.  $D_{22}(R_1R_2) = D_{22}(R_1)R_2 - R_1D_{22}(R_2)$  for any  $R_1, R_2 \in B(\mathcal{H}_2)$ . Hence  $D_{22}$  is a derivation. Since every derivation is an inner derivation on  $B(\mathcal{H}_2)$ , there exists an operator  $\tilde{D} \in B(\mathcal{H}_2)$  such that  $D_{22}(R) = R\tilde{D} - \tilde{D}R$  for any  $R \in B(\mathcal{H}_2)$ .

Step 3. We claim that  $A_{11}$  is a derivation and  $B_{12}(XV) = XB_{12}(V) + A_{11}(X)V$  for any operator  $X \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ .

For any invertible operator  $X \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ , taking  $Y = F - XV$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = X^{-1}E$ ,  $W = 0$  and  $R = I_{\mathcal{H}_2}$  in the equation (6), then we have

$$\begin{aligned} A_{12}(E) + B_{12}(F) &= A_{11}(X)V + B_{11}(F)V - B_{11}(XV)V + A_{12}(X) + B_{12}(F) \\ &\quad - B_{12}(XV) + XA_{12}(X^{-1}E) + XB_{12}(V) + XD_{12}(I_{\mathcal{H}_2}) \\ &\quad + FB_{22}(V) - XVB_{22}(V) + (F - XV)D_{22}(I_{\mathcal{H}_2}). \end{aligned}$$

Note that  $D_{22}(I_{\mathcal{H}_2}) = 0$  by  $D_{22}(R) = R\tilde{D} - \tilde{D}R$ . Replacing  $X$  and  $V$  by  $\lambda X$  and  $\mu V$  in the above equation, respectively, we have

$$\begin{aligned} A_{12}(E) &= \lambda\mu[A_{11}(X)V + XB_{12}(V) - B_{12}(XV)] + \mu[B_{11}(F)V + FB_{22}(V)] \\ &\quad - \lambda\mu^2[B_{11}(XV)V + XVB_{22}(V)] + \lambda[A_{12}(X) + XD_{12}(I_{\mathcal{H}_2})] + XA_{12}(X^{-1}E). \end{aligned}$$

The above equation implies that

$$B_{11}(XV)V + XVB_{22}(V) = 0 \tag{15}$$

and

$$B_{12}(XV) = XB_{12}(V) + A_{11}(X)V$$

for any invertible operator  $X \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ . Fixing an operator  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ , and define an linear mapping  $\varphi : B(\mathcal{H}_1) \rightarrow B(\mathcal{H}_2, \mathcal{H}_1)$  as follows

$$\varphi(X) = B_{12}(XV) - XB_{12}(V) - A_{11}(X)V$$

for any  $X \in B(\mathcal{H}_1)$ . Then  $\varphi(X) = 0$  for any invertible operator  $X \in B(\mathcal{H}_1)$ . It follows from Lemma 2.3 that  $\varphi \equiv 0$ , i.e.

$$B_{12}(XV) = XB_{12}(V) + A_{11}(X)V \quad (16)$$

for any operator  $X \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ .

Now we show that  $A_{11}$  is a derivation. For any  $X_1, X_2 \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ , we have

$$B_{12}(X_1X_2V) = X_1X_2B_{12}(V) + A_{11}(X_1X_2)V$$

and

$$\begin{aligned} B_{12}(X_1X_2V) &= X_1B_{12}(X_2V) + A_{11}(X_1)X_2V \\ &= X_1X_2B_{12}(V) + X_1A_{11}(X_2)V + A_{11}(X_1)X_2V \end{aligned}$$

The above two equation implies that  $[A_{11}(X_1X_2) - X_1A_{11}(X_2) - A_{11}(X_1)X_2]V = 0$ . So  $A_{11}$  is a derivation. Since every derivation is an inner derivation on  $B(\mathcal{H}_1)$ , there exists an operator  $A \in B(\mathcal{H}_1)$  such that  $A_{11}(X) = XA - AX$  for any  $X \in B(\mathcal{H}_1)$ .

Step 4. We claim that  $B_{22}(Y) = A_{21}(I_{\mathcal{H}_1})Y$  and  $B_{11}(Y) = YD_{21}(I_{\mathcal{H}_2})$  for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ , and  $D_{21}(R) = RD_{21}(I_{\mathcal{H}_2})$  for any  $R \in B(\mathcal{H}_2)$ .

1) For any  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Y = F - V$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E$ ,  $W = 0$  and  $R = I_{\mathcal{H}_2}$  in the equation (8), then we have

$$B_{22}(V) = A_{21}(I_{\mathcal{H}_1})V \quad (17)$$

for any  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ .

2) For any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $R \in B(\mathcal{H}_2)$ , taking  $X = \lambda I_{\mathcal{H}_1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = \lambda^{-1}E$ ,  $V = \lambda^{-1}(F - YR)$  and  $W = 0$  in the equation (6), then we have

$$\begin{aligned} A_{12}(E) + B_{12}(F) &= B_{11}(Y)F - B_{11}(Y)YR + \lambda A_{12}(I_{\mathcal{H}_1})R + B_{12}(Y)R \\ &\quad + A_{12}(E) + B_{12}(F) - B_{12}(YR) + \lambda D_{12}(R) \\ &\quad + \lambda^{-1}YB_{22}(F) - \lambda^{-1}YB_{22}(YR) + YD_{22}(R) \end{aligned}$$

i.e.

$$0 = \lambda^{-1}[B_{11}(Y)E + Y A_{21}(E)] + [Y D_{21}(R) - B_{11}(YR)] + \lambda D_{11}(R)$$

The above equation implies that

$$B_{11}(YR) = Y D_{21}(R).$$

In particular, the following equation holds.

$$B_{11}(Y) = Y D_{21}(I_{\mathcal{H}_2}) \quad (18)$$

for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ . It follows that  $Y D_{21}(R) = B_{11}(YR) = Y R D_{21}(I_{\mathcal{H}_2})$ . Thus we have

$$D_{21}(R) = R D_{21}(I_{\mathcal{H}_2}) \quad (19)$$

for any  $R \in B(\mathcal{H}_2)$ .

Step 5. We claim that  $C_{11}(W) = -A_{12}(I_{\mathcal{H}_1})W$  and  $C_{22}(W) = W A_{12}(I_{\mathcal{H}_1})$  for any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ , and  $A_{12}(X) = X A_{12}(I_{\mathcal{H}_1})$  for any  $X \in B(\mathcal{H}_1)$ .

For any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$  and  $X \in B(\mathcal{H}_1)$ , taking  $Y = 0$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = X^{-1}E$ ,  $V = X^{-1}F$  and  $R = 0$  in the equation (5), then we have

$$A_{11}(E) + B_{11}(F) = A_{11}(X)X^{-1}E + A_{12}(X)W + X A_{11}(X^{-1}E) + X B_{11}(X^{-1}F) + X C_{11}(W).$$

Replacing  $X$  by  $\lambda X$  in the above equation, then we have

$$A_{11}(E) + B_{11}(F) = A_{11}(X)X^{-1}E + X A_{11}(X^{-1}E) + X B_{11}(X^{-1}F) + \lambda[A_{12}(X)W + X C_{11}(W)].$$

The above equation implies that  $A_{12}(X)W = -X C_{11}(W)$ . In particular,

$$C_{11}(W) = -A_{12}(I_{\mathcal{H}_1})W \quad (20)$$



for any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ . Furthermore  $A_{12}(X)W = -XC_{11}(W) = XA_{12}(I_{\mathcal{H}_1})W$ . Hence

$$A_{12}(X) = XA_{12}(I_{\mathcal{H}_1}) \quad (21)$$

for any  $X \in B(\mathcal{H}_1)$ . By the equation (12), we have

$$YC_{22}(W) = A_{12}(YW) = YWA_{12}(I_{\mathcal{H}_1})$$

The above equation implies that

$$C_{22}(W) = WA_{12}(I_{\mathcal{H}_1}) \quad (22)$$

for any  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ .

Step 6. We claim that  $A_{21}(X) = A_{21}(I_{\mathcal{H}_1})X$  for any  $X \in B(\mathcal{H}_1)$ .

For any invertible operator  $X \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ , taking  $Y = F - XV$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = X^{-1}E$ ,  $W = 0$  and  $R = I_{\mathcal{H}_2}$  in the equation (8), then we have

$$B_{22}(XV) = A_{21}(X)V$$

for any  $X \in B(\mathcal{H}_1)$  and  $V \in B(\mathcal{H}_2, \mathcal{H}_1)$ . It follows from the equation (17) that  $A_{21}(X)V = B_{22}(XV) = A_{21}(I_{\mathcal{H}_1})XV$ . Thus we have

$$A_{21}(X) = A_{21}(I_{\mathcal{H}_1})X \quad (23)$$

for any  $X \in B(\mathcal{H}_1)$ .

Step 7. We claim that  $B_{12}(Y) = YD - AY$  and  $C_{21}(W) = WA - DW$  for some  $D \in B(\mathcal{H}_2, \mathcal{H}_1)$ .

For any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E - YW$ ,  $V = F$  and  $R = 0$  in the equation (5), then we have

$$\begin{aligned} 0 &= B_{11}(Y)E - B_{11}(Y)YW + A_{12}(I_{\mathcal{H}_1})W + B_{12}(Y)W - A_{11}(YW) \\ &\quad + C_{11}(W) + YA_{21}(E) - YA_{21}(YW) + YB_{21}(F) + YC_{21}(W) \end{aligned}$$

Replacing  $Y$  and  $W$  by  $\lambda Y$  and  $\lambda W$  in the above equation, we have

$$\begin{aligned} 0 &= \lambda[B_{11}(Y)E + A_{12}(I_{\mathcal{H}_1})W + C_{11}(W) + YA_{21}(E) + YB_{21}(F)] \\ &\quad + \lambda^2[B_{12}(Y)W - A_{11}(YW) + YC_{21}(W)] - \lambda^3[YA_{21}(YW) + B_{11}(Y)YW]. \end{aligned}$$

The above equation implies that

$$B_{12}(Y)W - A_{11}(YW) + YC_{21}(W) = 0.$$

Since  $A_{11}$  is a derivation and  $A_{11}(X) = XA - AX$ , the above equation implies that

$$Y[WA - C_{21}(W)] = [B_{12}(Y) + AY]W$$

for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ . By Theorem 2.2, there exists  $D \in B(\mathcal{H}_2)$  such that

$$C_{21}(W) = WA - DW \quad (24)$$

and

$$B_{12}(Y) = YD - AY \quad (25)$$

for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $W \in B(\mathcal{H}_1, \mathcal{H}_2)$ . On the other hand, for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $R \in B(\mathcal{H}_2)$ , taking  $X = I_{\mathcal{H}_1}$ ,  $Z = 0$ ,  $Q = 0$ ,  $U = E$ ,  $V = F - YR$  and  $W = 0$  in the equation (6), then we have

$$\begin{aligned} 0 &= B_{11}(Y)F - B_{11}(Y)YR + A_{12}(I_{\mathcal{H}_1})R + B_{12}(Y)R \\ &\quad - B_{12}(YR) + D_{12}(R) + YB_{22}(F) - YB_{22}(YR) + YD_{22}(R) \end{aligned}$$

Replacing  $Y$  and  $R$  by  $\lambda Y$  and  $\lambda R$  in the above equation, we have

$$0 = \lambda[B_{11}(Y)F + A_{12}(I_{\mathcal{H}_1})R + D_{12}(R) + YB_{22}(F)] \\ \lambda^2[B_{12}(Y)R - B_{12}(YR) + YD_{22}(R)] - \lambda^3[B_{11}(Y)YR + YB_{22}(YR)]$$

The above equation implies that

$$B_{12}(YR) = B_{12}(Y)R + YD_{22}(R)$$

for any  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$  and  $R \in B(\mathcal{H}_2)$ . It follows from the above equation and the equation (25) and  $D_{22}(R) = R\tilde{D} - \tilde{D}R$  that

$$YRD - AYR = B_{12}(YR) = B_{12}(Y)R + YD_{22}(R) = YDR - AYR + YR\tilde{D} - Y\tilde{D}R,$$

i.e.  $YR[D - \tilde{D}] = Y[D - \tilde{D}]R$ . Furthermore  $R[D - \tilde{D}] = [D - \tilde{D}]R$  for any  $R \in B(\mathcal{H}_2)$ . Hence  $D - \tilde{D} = \alpha I_{\mathcal{H}_2}$  for some  $\alpha \in \mathcal{C}$ . It is easy to verify that

$$D_{22}(R) = R\tilde{D} - \tilde{D}R = RD - DR \quad (26)$$

for any  $R \in B(\mathcal{H}_2)$ .

Step 8. In summary, we have

$$\begin{aligned} A_{11}(X) &= XA - AX, & B_{11}(y) &= YD_{21}(I_{\mathcal{H}_2}), \\ A_{12}(X) &= XA_{12}(I_{\mathcal{H}_1}), & B_{12}(y) &= YD - AY, \\ A_{21}(X) &= A_{21}(I_{\mathcal{H}_1})X, & B_{21}(y) &= 0, \\ A_{22}(X) &= 0, & B_{22}(y) &= A_{21}(I_{\mathcal{H}_1})Y, \\ \\ C_{11}(Z) &= -A_{12}(I_{\mathcal{H}_1})Z, & D_{11} &= 0, \\ C_{12}(Z) &= 0, & D_{12}(Q) &= -A_{12}(I_{\mathcal{H}_1})Q, \\ C_{21}(Z) &= ZA - DZ, & D_{21}(Q) &= QD_{21}(I_{\mathcal{H}_2}), \\ C_{22}(Z) &= ZA_{12}(I_{\mathcal{H}_1}), & D_{22} &= QD - DQ. \end{aligned}$$

Combining the equation (15) with  $B_{11}(Y) = YD_{21}(I_{\mathcal{H}_2})$  and  $B_{22}(Y) = A_{21}(I_{\mathcal{H}_1})Y$ , we have

$$0 = B_{11}(XY)Y + XYB_{22}(Y) = XYD_{21}(I_{\mathcal{H}_2})Y + XYA_{21}(I_{\mathcal{H}_1})Y$$

for any  $X \in B(\mathcal{H}_1)$  and  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ . Hence we have

$$D_{21}(I_{\mathcal{H}_2}) = -A_{21}(I_{\mathcal{H}_1}).$$

For the convenience, we write  $A_{12}(I_{\mathcal{H}_1}) = B$  and  $D_{21}(I_{\mathcal{H}_2}) = C$ . Then we have

$$\left\{ \begin{aligned} \varphi\left(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} XA - AX & XB \\ -CX & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right) &= \begin{bmatrix} YC & YD - AY \\ 0 & -CY \end{bmatrix} = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}\right) &= \begin{bmatrix} -BZ & 0 \\ ZA - DZ & ZB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix}, \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}\right) &= \begin{bmatrix} 0 & -BQ \\ QC & QD - DQ \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \end{aligned} \right.$$

for any  $X \in B(\mathcal{H}_1)$ ,  $Y \in B(\mathcal{H}_2, \mathcal{H}_1)$ ,  $Z \in B(\mathcal{H}_1, \mathcal{H}_2)$  and  $Q \in B(\mathcal{H}_2)$ , i.e.

$$\varphi\left(\begin{bmatrix} X & Y \\ Z & Q \end{bmatrix}\right) = \begin{bmatrix} X & Y \\ Z & Q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & Y \\ Z & Q \end{bmatrix}$$

for any  $\begin{bmatrix} X & Y \\ Z & Q \end{bmatrix} \in B(\mathcal{H})$ . Hence  $\varphi$  is an inner derivation. This completes the proof.  $\square$

**Theorem 3.1** *Let  $\mathcal{N}$  be a complete nest on Hilbert space  $\mathcal{H}$ . Then  $G \in \text{alg}\mathcal{N}$  is an all-derivable point if and only if  $G \neq 0$ .*

**Proof.** Supposing that  $\mathcal{N}$  is a nontrivial complete nest, this is directly the conclusion of Theorem 2.3 in [7]. Supposing that  $\mathcal{N}$  is a trivial complete nest, then  $\text{alg}\mathcal{N} = B(\mathcal{H})$ . This is directly the conclusion of Theorem 2.1. This completes the proof.  $\square$

## Reference

- 1 R.L. Crist, Local derivations on operator algebras, J. Funct. Anal. 135 (1996) 76-92.
- 2 W. Jing, S.J. Lu, P.T. Li, Characterisations of derivations on some operator algebras, Bull. Austral. Math. Soc. 66 (2002) 227-232.
- 3 J.C. Hou, X.F. Qi, Additive maps derivable at some points on J-subspace lattice algebras, Linear Algebra Appl. 429 (2008) 1851-1863.
- 4 D.R. Larson, A.R. Sourour, Local derivations and local automorphisms of  $\mathcal{B}(X)$ , Operator Algebras and Applications, Proc. Symp. Pure Math. 51 (1990) 187-194.
- 5 J.K. Li, Z.D. Pan, H. Xu, Characterizations of isomorphisms and derivations of some algebras. J. Math. Anal. Appl. 332 (2007) 1314-1322.
- 6 A.E. Taylor, D.C. Lay, Introduction to functional analysis(Second Edition), John Wiley and Sons, Inc. New York, 1980.
- 7 Y. F. Zhang, J.C. Hou, X.F. Qi, Characterizing derivations for any nest algebras on Banach space by their behaviors at an injective operator, Linear Algebra Appl., 449 (15) (2014) 312-333.
- 8 J. Zhu, S. Zhao, Characterizations all-derivable points in nest algebras, Proc. Amer. Math. Soc. 141(7) (2013) 2343-2350.